

ERGODIC PROPERTIES OF THE RANDOM WALK ADIC TRANSFORMATION OVER THE β -TRANSFORMATION

MICHAEL BROMBERG

SCHOOL OF MATHEMATICAL SCIENCES, TEL AVIV UNIVERSITY, TEL AVIV 69978, ISRAEL.

ABSTRACT. We define a random walk adic transformation associated to an aperiodic random walk on $G = \mathbb{Z}^k \times \mathbb{R}^{D-k}$ driven by a β -transformation and study its ergodic properties. In particular, this transformation is conservative, ergodic, infinite measure preserving and we prove that it is asymptotically distributionally stable and bounded rationally ergodic. Related earlier work appears in [AS] and [ANSS] for random walk adic transformations associated to an aperiodic random walk driven by a subshift of finite type.

1. INTRODUCTION

Let (X, \mathcal{B}, m) be a σ -finite measure space with infinite measure m and let $T : X \rightarrow X$ be a conservative, ergodic transformation preserving the measure m . It is a consequence of Hopf's ratio ergodic theorem that for every $f \in L^1(m)$, the normalized Birkhoff sums $\frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x)$ tend to 0 a.e. Moreover, (see [Aa1, Theroem 2.4.2]), for every sequence of constants $a_n > 0$, either $\liminf_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=0}^{n-1} f \circ T^k(x) = 0$ a.e. for all non-negative $f \in L^1(m)$ or there exists a subsequence n_k such that $\lim_{k \rightarrow \infty} \frac{1}{a_{n_k}} \sum_{j=0}^{n_k-1} f \circ T^j(x) = \infty$ a.e. for all non-negative $f \in L^1(m)$. It follows that there is no sequence of constants a_n , such that $\sum_{k=0}^{n-1} f \circ T^k \propto a_n$. Nevertheless, for certain transformations, there are weaker types of convergence for which $\frac{1}{a_n} \sum_{k=0}^{n-1} f \circ T^k$ converges. One such notion, which we proceed to define in the following paragraph and is the subject of study in this paper, is that of distributional stability (see [Aa1, 3.6]).

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Recall that convergence in distribution of a sequence of random variables f_n to a random variable f all taking values in some Polish space \mathcal{C} , means that $E(g \circ f_n) \xrightarrow{n \rightarrow \infty} E(g \circ f)$ for all bounded, continuous $g : \mathcal{C} \rightarrow \mathbb{R}$. Let (X, \mathcal{B}, m) be a σ -finite, infinite measure space, $f_n : X \rightarrow [0, \infty]$ be measurable and let $f \in [0, \infty]$ be a random variable defined on some probability space. Let ν be some probability measure absolutely continuous with respect to m . Then $\{f_n\}$ may be viewed as a sequence of random variables defined on the probability space (X, \mathcal{B}, ν) and we write $f_n \xrightarrow{\nu} f$ if f_n converges in distribution to f . We say that f_n converges strongly in distribution to f and write $f_n \xrightarrow{\mathcal{L}(m)} f$ if $f_n \xrightarrow{\nu} f$ with respect to any probability measure ν , absolutely continuous with respect to m . Equivalently, this means that $g \circ f_n \rightarrow E(g(f))$ weak-* in $L^\infty(m)$ for each bounded and continuous $g : [0, \infty] \rightarrow \mathbb{R}$, i.e. $\int g \circ f_n \cdot p \, dm \rightarrow E(g \circ f) \int p \, dm$ for all $p \in L^1(m)$ (here and throughout this paper, the space $[0, \infty]$ is the one point compactification of $[0, \infty)$).

Definition 1. A conservative, ergodic measure preserving transformation (X, \mathcal{B}, m, T) is distributionally stable if there is a sequence of constants $a_n > 0$, and a random variable Y taking values in $(0, \infty)$, such that

$$(1.1) \quad \frac{1}{a_n} S_n(f) \xrightarrow{\mathcal{L}(m)} Y m(f)$$

for all $f \in L^1(m)$, $f \geq 0$, where $S_n(f) := \sum_{k=0}^{n-1} f \circ T^k$ and $m(f) := \int_X f \, dm$.

Note that by Hopf's rational ergodic theorem if (1.1) holds for some $f \in L^1(m)$, $f \geq 0$ then it holds for all $f \in L^1(m)$. Moreover, if (1.1) holds, then the sequence a_n is unique up to asymptotic equality and is called the return sequence of a_n . In [Aa2] distributional stability was proved for pointwise dual ergodic transformations having regularly varying return sequences with Mittag-Leffler distributions appearing as limits (see also [Aa2]). More recently, (see [AS], [ADDS]) distributional stability was proved for certain transformations with exponential chi-squared distributions appearing as limits. In particular, it is proved in [AS] that the random walk adic transformation associated with an aperiodic random walk on $\mathbb{Z}^k \times \mathbb{R}^{D-k}$ driven by

a subshift of finite type is distributionally stable with exponential chi squared distribution (with D degrees of freedom) in the limit. One purpose of this paper is to define a random walk adic transformation associated with an aperiodic walk driven by the β -transformation, and to prove that it is distributionally stable with chi squared exponential distribution in the limit (see theorem 24).

Another notion which we study in this paper is that of bounded rational ergodicity.

Definition 2. A conservative, ergodic, infinite measure preserving transformation (X, \mathcal{B}, m, T) is called *bounded rationally ergodic* (see [Aa3]) if there exists a measurable set $A \subseteq X$ with $0 < m(A) < \infty$, such that there exists $M > 0$, such that for all $n \geq 1$

$$(1.2) \quad \left\| \sum_{k=0}^{n-1} \mathbb{1}_A \circ T^k \right\|_{L^\infty(A)} \leq M \int_A \left(\sum_{k=0}^{n-1} \mathbb{1}_A \circ T^k \right) dm.$$

The rate of growth of the sequence $a_n = \frac{1}{m(A^2)} \int_A \left(\sum_{k=0}^{n-1} \mathbb{1}_A \circ T^k \right) dm$ does not depend on a set A satisfying (1.2). Bounded rational ergodicity implies a kind of absolutely normalized ergodic theorem stating that

$$\frac{S_n(f)}{a_n} \rightsquigarrow \int_X f dm, \forall f \in L^1(m)$$

where $f_n \rightsquigarrow f$ means that $\forall m_l \uparrow \infty \exists n_k = m_{l_k} \uparrow \infty$ such that $\forall p_j = n_{k_j}$ such that $\forall p_j = n_{k_j} \uparrow \infty$, we have $\frac{1}{N} \sum_{j=1}^N f_{p_j} \xrightarrow{N \rightarrow \infty} f$ a.e. In section (6) we prove that the random walk adic transformation associated with an aperiodic random walk driven by the β -transformation is bounded rationally ergodic with $a_n \propto \frac{n}{\sqrt{\log n}}$. Bounded rational ergodicity of random walk adic transformations associated to an aperiodic random walk driven by a subshift of finite type are studied in [ANSS].

2. ADIC TRANSFORMATION ASSOCIATED WITH A β -TRANSFORMATION

2.1. β -Transformations. In this section we give some preliminaries concerning β -transformations and refer the reader to [DK, Pa, Re, Bl] for proofs of all facts stated herein.

In what follows $[x] := \min \{n \in \mathbb{Z} : n \leq x\}$, $(x) = x - [x]$, λ is the Lebesgue measure on $[0, 1)$.

The beta transformation is defined on $X = [0, 1)$ by

$$T_\beta(x) := \beta x \bmod 1 \equiv (\beta x)$$

It was proved by Rényi (see [Re]) that there exists a unique, ergodic, T_β -invariant measure m , equivalent to the Lebesgue measure on X . Moreover, the invariant density, which we denote by h , satisfies $1 - \frac{1}{\beta} \leq h(x) \leq \frac{1}{1-\frac{1}{\beta}}$ for all $x \in X$. Every $x \in X$ has a β -expansion of the form $x = \sum_{k=1}^{\infty} \frac{d_k(x)}{\beta^k}$ where $d_k(x) := \lfloor \beta T_\beta^{k-1} x \rfloor$. Although 1 is not in the domain of T , defining $T(1) = \beta - [\beta]$, we can still consider the β -expansion of 1 given by $1 = \sum_{k=1}^{\infty} \frac{d_k(1)}{\beta^k}$, where $d_k(1) := \lfloor \beta T_\beta^{k-1} 1 \rfloor$. In what follows, we denote by $d(x, \beta)$ the sequence of digits in the β -expansion of $x \in X \cup \{1\}$, i.e $d(x, \beta) = (d_1(x), d_2(x), \dots)$. Not every sequence of integers between 1 and $[\beta]$ gives rise to a β -expansion of some $x \in X$. We say that the sequence $(d_1, d_2, \dots) \in \{0, \dots, [\beta]\}^{\mathbb{Z}}$ is β -admissible if it is the β -expansion of some $x \in X$, i.e if $d_k = d_k(x)$ for some $x \in X$. The set of β -admissible sequences, which we denote by S_β , is a closed shift invariant subspace of $\{0, \dots, [\beta]\}^{\mathbb{Z}}$. This set may be linearly ordered by lexicographic order which we denote by \prec_{lex} in the obvious way. Namely, for distinct $\omega, \eta \in S_\beta$,

$$\omega \prec_{lex} \eta$$

if there exists $n \in \mathbb{N}$ such that $\omega_i = \eta_i$ for all $i < n$ and $\omega_n < \eta_n$.

The map

$$(2.1) \quad \psi(x) = (d_1(x), d_2(x) \dots)$$

is one to one, onto, bi-measurable from X to S_β and satisfies $\psi^{-1} \circ \sigma \circ \psi(x) = T_\beta(x)$ for all $x \in X$, where σ is the left shift on S_β . Thus, ψ is an isomorphism between the systems $(X, \mathcal{B}, m, T_\beta)$ and $(S_\beta, \mathcal{C}, \nu, \sigma)$ where \mathcal{C} is the natural (Borel) σ -algebra on $\{0, \dots, [\beta]\}^{\mathbb{Z}}$ restricted to S_β , and ν is the push forward of m by ψ . If the β -expansion of 1 is finite then (S_β, σ) is a subshift of finite type and as the case of subshifts of finite types has been dealt with in [AS], we shall assume that $d(1, \beta)$ is not finite. In this case, the set of admissible sequences is identified by the following theorem [Pa]:

Theorem 3. *If $d(1, \beta)$ is not eventually periodic then the sequence $\omega = (d_1, d_2, \dots)$ is β -admissible if and only if $\sigma^n \omega \prec_{lex} d(1, \beta)$ for all $n \in \mathbb{N}$.*

Thus, in the case that $d(1, \beta)$ is not eventually periodic

$$S_\beta = \left\{ \omega \in \{0, \dots, [\beta]\}^{\mathbb{Z}} : \sigma^n \omega \prec_{lex} d(1, \beta) \right\}.$$

Remark 4. Henceforth, we assume that $d(1, \beta)$ is not eventually periodic.

Let $[d_1, \dots, d_k] := \left\{ x \in X : x = \sum_{i=1}^{\infty} \frac{d_i(x)}{\beta^k}, d_i = d_i(x), i = 1, \dots, k \right\}$. $[d_1, \dots, d_k]$ is called a cylinder of rank k . A cylinder $[d_1, \dots, d_k]$ is called a full cylinder of rank k if

$$\lambda(T^k[d_1, \dots, d_k]) = 1$$

and non-full otherwise. All full cylinders Δ_k of rank k satisfy $\lambda(\Delta_k) = \frac{1}{\beta^k}$ (see [DK]) and therefore have equal Lebesgue measure. We state the following lemma for future reference.

Lemma 5. [DK] *Given any $k \in \mathbb{N}$, X can be covered by disjoint full intervals of rank k or $k + 1$.*

2.2. Adic transformation of the β -transformation. The purpose of this section is to define the adic transformation of the β -transformation. Adic transformations appear in [Ve], where they are defined over Bratelli diagrams. We briefly describe the construction. Let

$\mathcal{S}_k = \{0, \dots, a_k\}$ be a sequence of finite alphabets and let $A_k : \mathcal{S}_k \times \mathcal{S}_{k+1} \rightarrow \{0, 1\}$ be a sequence of transition matrices. Define $\Omega := \prod_{i=1}^{\infty} \mathcal{S}_i$ and

$$\Sigma := \{\omega \in \Omega : A_k(\omega_k, \omega_{k+1}) = 1\}.$$

The adic transformation over the Bratelli diagram $\{\mathcal{S}_k, A_k\}$ assigns to $\omega \in \Sigma$ the element of Σ that succeeds ω in the reverse lexicographic order (see definition (7) below). The adic transformation over the β -transformation will be defined in a similar way, with the exception that the β -transformation is not a Bratelli diagram since the set of allowable digits appearing in the n th place of the β -expansion of a number $x \in X$ depends on the whole prefix and not only on the preceding digit in the expansion.

Definition 6. The tail relation of the β -transformation is the equivalence relation on X given by

$$\begin{aligned} \mathcal{T}(T_\beta) : &= \{(x, y) \in X \times X : \exists K \in \mathbb{N} \text{ such that } d_k(x) = d_k(y) \ \forall k \geq K\} \\ &= \bigcup_{n \geq 0} \{(x, y) : T_\beta^n x = T_\beta^n y\} \end{aligned}$$

Definition 7. The reverse lexicographic order on X is the partial order \prec_{rev} defined by $x \prec_{rev} y$ if and only if there exists $n \in \mathbb{N}$, such that $d_k(x) = d_k(y)$ for all $k > n$ and $d_n(x) < d_n(y)$.

Thus the equivalence sets of $\mathcal{T}(T_\beta)$ are linearly ordered by \prec_{rev} .

The adic transformation $\tau : X \rightarrow X$ of T_β is that transformation which parametrizes the tail relation on X (in the sense that $x \sim_{\mathcal{T}(X)} y$ if and only if $y = \tau^n x$ for some $n \in \mathbb{Z}$) and assigns to each x the minimal y that satisfies $y \succ_{rev} x$. Thus τ is defined by $\tau(x) := \min\{y : d(y, \beta) \succ_{rev} d(x, \beta)\}$ where the minimum is taken with respect to \prec_{rev} .

Our next objective is to identify the set on which τ is well defined, and on which τ is invertible. To this purpose, we identify the set of maximal point of X with respect to the reverse lexicographic order, and show that τ is well defined outside this set.

Proposition 8. *Let*

$$\Sigma_{max} := \left\{ x : T_{\beta}^n x \geq 1 - \frac{1}{\beta}, \forall n \in \mathbb{N}_* \right\} = \left\{ x : \forall n \in \mathbb{N}_* x \in T^{-n} [1 - \beta, 1) \right\}.$$

Then τ is defined for all $x \in X \setminus \Sigma_{max}$ and is not defined on Σ_{max} . Moreover, if for $x \in X \setminus \Sigma_{max}$

$$d(x, \beta) = (d_1(x), d_2(x), \dots)$$

is the sequence of digits in the β -expansion of x , then

$$\tau(x) = ([0]_{n_0}, d_{n_0+1}(x) + 1, d_{n_0+2}(x), d_{n_0+3}(x), \dots)$$

where $n_0 := \min \left\{ n \in \mathbb{N}_ : T_{\beta}^n x < 1 - \frac{1}{\beta} \right\}$ and $[0]_n := 0, \dots, 0$.*
 $n \text{ times}$

Proof. Let $x \in X \setminus \Sigma_{max}$, $n_0 := \min \left\{ n : T_{\beta}^n x < 1 - \frac{1}{\beta} \right\}$ and let $d(x, \beta) = (d_1(x), d_2(x), \dots)$ be the β -expansion of x . Since $\psi^{-1} \circ \sigma \circ \psi(x) = T_{\beta}(x)$ where ψ is as in (2.1), it follows that the β -expansion of $T_{\beta}^{n_0} x$ is $(d_{n_0+1}(x), d_{n_0+2}(x), \dots)$. Now, note that if $y \in X$, $y + \frac{1}{\beta} < 1$ then

$$d\left(y + \frac{1}{\beta}, \beta\right) = (d_1(y) + 1, d_2(y), d_3(y), \dots).$$

This is seen as follows: the first digit of the β -expansion of $y + \frac{1}{\beta}$ is

$$\left[\beta \left(y + \frac{1}{\beta} \right) \right] = [\beta y + 1] = [\beta y] + 1,$$

while the remaining digits are formed by the expansion of $y + \frac{1}{\beta} - \frac{[\beta y] + 1}{\beta} = y - \frac{[\beta y]}{\beta}$, which in turn, coincide with the remaining digits in the expansion of y . Since $T_{\beta}^{n_0} x + \frac{1}{\beta} < 1$, it follows

that

$$d\left(T_{\beta}^{n_0}x + \frac{1}{\beta}, \beta\right) = (d_{n_0+1}(x) + 1, d_{n_0+2}(x), d_{n_0+3}(x), \dots)$$

is an admissible sequence and therefore, $\omega = ([0]_{n_0}, d_{n_0+1}(x) + 1, d_{n_0+2}(x), d_{n_0+3}(x), \dots)$ is also admissible ($\sigma^n \omega \prec_{lex} d(1, \beta)$ for $n < n_0$ since the first digit in $d(1, \beta)$ is $[\beta] > 0$, and $\sigma^n \omega \prec_{lex} d(1, \beta)$ for $n \geq n_0$ since $\sigma^{n_0} \omega$ is admissible). Let $y \in X$ be such that

$$d(y, \beta) = \omega.$$

Obviously $x \prec_{rev} y$, and by definition of the reverse lexicographic order there must be only finite number of elements that lie strictly between x and y . This implies that the set

$$\{y : d(y, \beta) \succ_{rev} d(x, \beta)\}$$

has a minimal element and therefore $\tau(x)$ is defined.

We show that $\tau(x) = y$. If not, then there must exist $z \in X$, such that $x \prec_{rev} z \prec_{rev} \omega$. This implies that there exists $k \leq n_0$ such that $d_k(x) < d_k(y)$ and $d_n(x) = d_n(y)$ for all $n > k$. This implies that

$$\sigma^{k-1}(d(y, \beta)) \succeq_{lex} (d_k(x) + 1, d_{k+1}(x), d_{k+2}(x), \dots),$$

which in this case means that $(d_k(x) + 1, d_{k+1}(x), d_{k+2}(x), \dots)$ is an admissible sequence. Therefore, $\frac{d_k(x)+1}{\beta} + \sum_{i=1}^{\infty} \frac{d_{k+i}(x)}{\beta^{i+1}} < 1$ and it follows that

$$T_{\beta}^{k-1}x = \sum_{i=0}^{\infty} \frac{d_{k+i}(x)}{\beta^{i+1}} < 1 - \frac{1}{\beta}$$

which contradicts the definition of n_0 .

Note that in particular, the last argument shows that if $x \prec_{lex} z$, then there must be some $k \in \mathbb{N}_*$, such that $T_{\beta}^k x < 1 - \frac{1}{\beta}$ and therefore, $x \notin \Sigma_{max}$. This completes the proof. \square

Corollary 9. *There exists a measurable, τ invariant set $\hat{X} \subseteq X$ with $\lambda(\hat{X}) = 1$ restricted to which τ is invertible.*

Proof. Using proposition 8 inductively, we conclude that $\tau^n x$ is defined for all $n \in \mathbb{N}$, if and only if $T_\beta^n x + \frac{1}{\beta} < 1$ for infinitely many $n \in \mathbb{N}$. Letting Σ_{max} be as in proposition 8, we have the equality

$$\left\{ x \in X : T_\beta^n x + \frac{1}{\beta} < 1 \text{ for finitely many } n \in \mathbb{N}_* \right\} = \bigcup_{k \in \mathbb{N}_*} T_\beta^{-k} \Sigma_{max}.$$

Thus, all powers of τ are well-defined for all $x \in Y := X \setminus \bigcup_{k \in \mathbb{N}_*} T_\beta^{-k} \Sigma$ and there is some power of τ which is undefined for $x \notin Y$. It follows that Y is τ -invariant.

We show that Y has full Lebesgue measure. Since T_β is ergodic and $m([1 - \beta, 1)) < 1$, it follows by Birkhoff's ergodic theorem that there exists $\rho < 1$ such that for m almost every x , and large enough n , $\frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{[0, 1-\beta)} < \rho$. This shows that $m(\Sigma) = 0$ and therefore, $m\left(\bigcup_{k \in \mathbb{N}_*} T_\beta^{-k} \Sigma\right) = 0$. By equivalence of the Lebesgue measure and the measure m , it follows that $\lambda(Y) = 1$.

Setting $\hat{X} := Y \setminus \bigcup_{k=0}^{\infty} T_\beta^{-k}(\bar{0})$, where $\bar{0} = (0, 0, \dots)$ we obtain a set of full Lebesgue measure, invariant under τ (invariance is seen by the fact that τ parametrizes the tail relation, i.e x and τx must be in the same equivalence class). Since it is clear from the definition of τ that it is an injective map, to prove invertibility, it suffices to show that $\tau : \hat{X} \rightarrow \hat{X}$ is onto. Let $x \in \hat{X}$ and let $d(x, \beta) = (d_1(x), d_2(x), \dots)$ be its expansion. Let $n_0 := \min\{k : d_k(x) > 0\}$. Then $\omega = (d_1(x), \dots, d_{n_0}(x) - 1, d_{n_0+1}(x), \dots)$ is an admissible sequence if $y \in X$ has expansion ω , then $y \prec_{rev} x$ and there are finitely many elements between y and x ordered by the reverse lexicographic order. It follows that $\tau^k x = y$ for some $k \in \mathbb{N}$, and that $y \in \hat{X}$. Therefore, $\tau : \hat{X} \rightarrow \hat{X}$ is onto and the proof is complete. \square

Proposition 10. $\tau : \hat{X} \rightarrow \hat{X}$ preserves the Lebesgue measure.

Proof. Let $x \in \hat{X}$. Since the set of all cylinders $\{\Delta_n\}_{n=1}^\infty$ generates \mathcal{B} , it suffices to prove that there exists a cylinder Δ_n such that $x \in \Delta_n$, and $\lambda(\tau(\Delta_n)) = \lambda(\Delta_n)$. Let

$$n_0 = \min \left\{ T^n x + \frac{1}{\beta} < 1 \right\}.$$

Then by proposition 8

$$\tau x = ([0]_{n_0}, d_{n_0+1}(x) + 1, d_{n_0+2}(x), d_{n_0+3}(x), \dots).$$

Fix $n_1 > n_0 + 1$. By lemma 5, there exist two cylinders A, B of rank n_1 or $n_1 + 1$ such that $x \in A, \tau x \in B$. If the ranks are different, by concatenating the last symbol of the longer cylinder to the shorter cylinder we obtain two full cylinders of equal rank. Doing this will not change the fact that $x \in A$ and $\tau x \in B$, because the above formula for τx shows that the digits in the expansions of x and τx coincide for the index $n_1 + 1$. Therefore, without loss of generality, we may assume that both A and B are full cylinders of rank $\tilde{n} > n_0 + 1$. It follows that

$$A = (d_1(x), d_2(x), \dots, d_{\tilde{n}}(x))$$

and

$$B = ([0]_{n_0}, d_{n_0+1}(x) + 1, d_{n_0+2}(x), \dots, d_{\tilde{n}}(x)).$$

Therefore, proposition 8 shows that $y \in A \implies \tau y \in B$ and it is easy to see by definition of the reverse lexicographic order that $\tau^{-1}(B) = A$. Thus, $\tau(A) = B$ and since full intervals of equal rank have same Lebesgue measure the claim follows. \square

3. RANDOM WALK ADIC TRANSFORMATION ASSOCIATED WITH AN APERIODIC RANDOM WALK FOR THE BETA TRANSFORMATION.

Let $G = \mathbb{R}$ or $G = \mathbb{Z}$ and let $\varphi : X \rightarrow G$. The random walk over the β -transformation generated by f is the skew product

$$(X \times G, \mathcal{B}(X) \times \mathcal{B}(G), \tilde{m}, \sigma_\varphi)$$

where $\tilde{m} := m \times dy$, dy is the Haar measure on G and $\sigma_\varphi(x, y) = (T_\beta x, y + \varphi(x))$. In what follows, Birkhoff sums of the form $\sum_{k=0}^{n-1} \varphi(T_\beta^k x)$ will be denoted by $\varphi_n(x)$.

Similarly to how the adic transformation τ parametrizes the tail relation of T_β , the random walk adic transformation associated to σ_φ is the (unique) skew product over $(\hat{X}, \mathcal{B} \cap \hat{X}, \lambda)$, which parametrizes the tail relation of σ_φ . To identify this note that the tail relation of σ_φ is given by

$$\mathcal{T}(\sigma_\varphi) = \{(x, y) \times (x', y') : (x, x') \in \mathcal{T}(T_\beta), \exists n_0 \forall n > n_0 \ y + \varphi_n(x) = y' + \varphi_n(x')\}.$$

Now let $(x, y) \times (x', y') \in \mathcal{T}(\sigma_\varphi)$. Since $(x, x') \in \mathcal{T}(T_\beta)$, it follows that there exists n such that $T_\beta^{n-1}(x) = T_\beta^{n-1}(y)$. Therefore,

$$y + \varphi_k(x) = y' + \varphi_k(x')$$

for all k greater than some $K \in \mathbb{N}$, if and only if $y + \varphi_n(x) = y' + \varphi_n(x')$. It follows that

$$(x, y) \times (x', y') \in \mathcal{T}_{\sigma_\varphi}$$

if and only if

$$(x, x') \in \mathcal{T}(T_\beta)$$

and

$$y' = y + \psi(x, x'),$$

where $\psi(x, x') = \sum_{k=0}^{\infty} \varphi(T_{\beta}^k x) - \varphi_k(T_{\beta}^k x')$. It follows that for $(x, y) \in \hat{X} \times \mathbb{R}^d$,

$$(\tau x, y + \phi(x)) \sim_{\mathcal{T}(\sigma_{\varphi})} (x, y)$$

if and only if $\phi(x) := \psi(x, \tau x)$. Thus, we define the random walk adic transformation as follows.

Definition 11. The random walk adic transformation associated to σ_{φ} is the skew product

$$\left(\hat{X} \times G, \left(\mathcal{B} \cap \hat{X} \right) \times \mathcal{B}(G), \mu, \tau_{\varphi} \right),$$

where $\mu = \lambda \times dy$, λ is the Lebesgue measure on X restricted to \hat{X} , dy is the Haar measure on G and

$$\tau_{\varphi}(x, y) = (\tau x, y + \phi(x))$$

where

$$\phi(x) := \psi(x, \tau x) = \sum_{k=0}^{\infty} \varphi(T^k x) - \varphi(T^k(\tau x)).$$

Note that since τ is invertible on \hat{X} , τ_{φ} is invertible and by the arguments above, for $(x, y), (x', y') \in \hat{X} \times \mathbb{R}^d$, $(x, y) \sim_{\mathcal{T}(\sigma_{\varphi})} (x', y')$ if and only if $\tau_{\varphi}^n(x, y) = (x', y')$ for some $n \in \mathbb{Z}$.

Denote by \hat{G} the dual group of G .

Definition 12. A measurable function $\varphi : X \rightarrow G$ is aperiodic if the only solutions the equation $\gamma \circ \varphi = \frac{\lambda g}{g \circ T}$ m -a.e, with $\gamma \in \hat{G}$, $|\lambda| = 1$ and a measurable $g : X \rightarrow \mathbb{S}^1$ are $\gamma \equiv 1$, $\lambda = 1$ and g is an a.e constant function.

We say that the random walk over the β -transformation is aperiodic, if it is generated by an aperiodic function φ . Aperiodicity is crucial for proving exactness and local limit theorems for the skew product $(X \times \mathbb{R}, \mathcal{B}(X) \times \mathcal{B}(G), \tilde{m}, \sigma_{\varphi})$. In order for these to hold, in addition to aperiodicity, we must make further regularity assumptions on the function φ , namely we need to restrict φ to a Banach space, on which the associated transfer operator (also known as

the Ruelle-Frobenius-Perron operator) acts quasi-compactly. This is the goal of the following section. All relevant definitions are provided therein.

4. ASSUMPTIONS ON THE OBSERVABLE φ AND IMPLICATIONS

The results of this section appear in [ADSZ] where they are proved in a more general context of piecewise monotonic, expanding maps of the interval. We list the results relevant to our case.

For an interval $A \subseteq X$, and $f : A \rightarrow \mathbb{R}$, define the variation of f on A to be $var_f(A) := \sup \sum_i |f(x_i) - f(x_{i-1})|$ where the supremum is taken over all finite partitions

$$x_1 < x_2 < \dots < x_n$$

of A . For $f \in L^1(m)$ set

$$\bigvee_A f := \inf \{var_{f^*}(A) : f^* = f \text{ a.e.}\}.$$

For $f \in L^\infty(m)$, define

$$\|f\|_{BV} := \|f\|_\infty + \bigvee_X f$$

and let

$$BV := \{f \in L^\infty(m) : \|f\|_{BV} < \infty\}$$

The space BV endowed with the norm $\|\cdot\|_{BV}$ is a Banach space.

We will also be interested in functions of bounded variation on each element of the natural partition of the unit interval for the β -transformation. This partition corresponds to the partition $\{[1], \dots, [\beta]\}$ of the associated β -shift and is given by

$$\alpha = \left\{ \left[0, \frac{1}{\beta}\right), \left[\frac{1}{\beta}, \frac{2}{\beta}\right), \dots, \left[\frac{[\beta]}{\beta}, 1\right) \right\}.$$

We say that $\varphi : X \rightarrow \mathbb{R}$ is locally of bounded variation on α , if

$$C_{\varphi, \alpha} := \sup_{A \in \alpha} \bigvee_A \varphi < \infty.$$

Note that since α is a finite partition, $C_{\varphi, \alpha} < \infty$ implies that φ is bounded.

Recall that for a non-singular dynamical system (Y, \mathcal{C}, μ, T) the transfer operator is an operator $\hat{T} : L^1(\mu) \rightarrow L^1(\mu)$, uniquely defined by the equality

$$\int_X f \circ \hat{T} \cdot g \, d\mu = \int_X f \cdot g \circ T \, d\mu$$

for every $f \in L^1(m)$, $g \in L^\infty(m)$. Let \hat{T}_β be the transfer operator of $(X, \mathcal{B}, m, T_\beta)$.

In what follows we also need the transfer operator $\mathcal{L} : L^1(m) \rightarrow L^1(m)$, defined by

$$(\mathcal{L}f)(x) = \sum_{y \in T_\beta^{-1}(x)} f(y).$$

Note that $\mathcal{L}f(x)$ is finite for almost every $x \in X$, and $\mathcal{L}f = \mathcal{L}\tilde{f} \pmod{m}$ if $f = \tilde{f} \pmod{m}$. The operator \mathcal{L} is also referred to as the transfer operator (or the Ruelle-Frobenius-Perron operator) and may be used to obtain the T_β invariant density h (see [Wa]). We have that β is an eigenvalue of \mathcal{L} corresponding to the function h , i.e $\mathcal{L}h = \beta h$ and the operator \mathcal{L} and \hat{T} are related by (see [Wa, Lemma 11])

$$\mathcal{L}(f) = \beta h \cdot \hat{T} \left(\frac{f}{h} \right) \quad \forall f \in L^1(m).$$

Definition 13. An operator G on a Banach space B is called quasi-compact with s dominating simple eigenvalues if

- (1) There exist G -invariant spaces F and H such that F is an s dimensional space and $B = F \oplus H$.
- (2) G is diagonalizable when restricted to F with all eigenvalues having modulus equal to the spectral radius of G , denoted by $\rho(G)$.

(3) When restricted to H , the spectral radius of G is strictly less than $\rho(G)$.

Definition 14. The fact that \hat{T}_β is a quasi-compact operator on BV with one simple dominating eigenvalue 1 and corresponding eigenspace of constant functions is proved in [ADSZ]. Thus, \hat{T}_β has the form $\hat{T}_\beta = m(f) \mathbb{1} + Q$ where the spectral radius of $Q : \mathcal{B} \rightarrow \mathcal{B}$ satisfies $\rho(Q) < 1$ and $m \circ Q = Q \mathbb{1} = 0$.

The characteristic function operator associated to a measurable function $\varphi : X \rightarrow \mathbb{R}$ is a family of operators $P(t) : L^1(m) \rightarrow L^1(m)$, $t \in \mathbb{R}$ defined by

$$P(t)f = \hat{T}_\beta(e^{it\varphi}f).$$

Let $\varphi : X \rightarrow \mathbb{R}$ be such that $C_{\varphi,\alpha} < \infty$. Then for $t \in \mathbb{R}$, $P(t) : BV \rightarrow BV$ is quasi-compact and $P(t)$ is twice continuously differentiable as a function from \mathbb{R} to $Hom(BV, BV)$. It follows from operator perturbation theory (see [ADSZ]) that there exists a δ neighborhood of 0, such that for $|t| < \delta$, $P(t)$ is quasi-compact with a simple dominating eigenvalue $\lambda(t)$, where $\lambda(t)$ has Taylor's expansion at 0 of the form

$$\lambda(t) = 1 + im(\varphi) - \sigma^2 t^2 + o(t^2), \quad \sigma \geq 0.$$

Moreover, $\sigma^2 = \lim_{n \rightarrow \infty} \frac{1}{n} Var_m(\phi_n)$ (here $Var(\varphi_n)$ denotes the variance of the sum φ_n) and $\sigma^2 = 0$ if and only if φ is a coboundary, i.e of the form $\varphi = f \circ T_\beta - f$ for some measurable $f : X \rightarrow \mathbb{R}$. As a consequence of that, exactness, conditional central limit theorems and conditional local theorems for the skew product $(X \times G, \mathcal{B}(X) \times \mathcal{B}(G), \tilde{m}, \sigma_\varphi)$ can be obtained (see [ADSZ], [AD], [HH]). We list these results.

Through the rest of this paper we assume that $\varphi : X \rightarrow G$, $G = \mathbb{R}$ or $G = \mathbb{Z}$, $C_{\varphi,\alpha} < \infty$, φ is aperiodic and $\lim_{n \rightarrow \infty} \frac{1}{n} Var_m(\varphi_n) = \sigma^2 > 0$.

Recall that a non-singular transformation on a standard probability space (Y, \mathcal{C}, μ, T) is exact if the tail σ -field of T defined by $\mathcal{T}(T) := \bigcap_{n=1}^{\infty} T^{-n}\mathcal{C}$ is trivial, i.e $\mathcal{T}(T) = \{\emptyset, Y\}$.

Theorem 15. [ADSZ, Theorem 7] *If $\varphi : X \rightarrow G$ where $G = \mathbb{R}$ or $G = \mathbb{Z}$ is aperiodic and $C_{\varphi, \alpha} < \infty$, then the skew product $(X \times G, \mathcal{B}(X) \times \mathcal{B}(G), \tilde{m}, \sigma_\varphi)$ is exact.*

Corollary 16. *If $\varphi : X \rightarrow G$ is aperiodic and $C_{\varphi, \alpha} < \infty$ then the random walk adic transformation $(\hat{X} \times G, (\mathcal{B} \cap \hat{X}) \times \mathcal{B}(G), \mu, \tau_\varphi)$ is conservative and ergodic.*

Proof. Ergodicity follows from exactness of the skew product $(X \times G, \mathcal{B}(X) \times \mathcal{B}(G), \tilde{m}, \sigma_\varphi)$. Indeed, since τ_φ parametrizes the tail relation of $(X \times G, \mathcal{B}(X) \times \mathcal{B}(G), \tilde{m}, \sigma_\varphi)$ any τ_φ invariant subset must be in the tail σ -field of σ_φ . Conservativity follows since τ_φ is invertible and ergodic (see [Aa1, Proposition 1.2.1]). \square

Theorem 17. [ADSZ, Theorem 9(1)] (CLT) *For an interval $I \subseteq \mathbb{R}$,*

$$\hat{T}_\beta^n \left(\mathbb{1}_{\left\{ \frac{\varphi_n}{\sigma\sqrt{n}} \in I \right\}} \right) (x) \xrightarrow{n \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_I e^{-\frac{t^2}{2}} dt,$$

uniformly in $x \in X$. In particular

$$m \left(\mathbb{1}_{\left\{ \frac{\varphi_n}{\sigma\sqrt{n}} \in I \right\}} \right) \longrightarrow \frac{1}{\sqrt{2\pi}} \int_I e^{-\frac{t^2}{2}} dt$$

.

Remark 18. Aperiodicity of φ is not required for the CLT.

Theorem 19. [ADSZ, Theorem 9(2)] (LLT - Discrete version) *Assume that $\varphi : X \rightarrow \mathbb{Z}$ is aperiodic. Then*

$$\sigma\sqrt{n}\hat{T}_\beta^n \left(\mathbb{1}_{\{\varphi_n = k_n\}} \right) (x) \xrightarrow{n \rightarrow \infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}, \quad k_n \in \mathbb{Z}, \quad \frac{k_n - nE_m(\varphi)}{\sigma\sqrt{n}} \rightarrow t$$

uniformly in $x \in X$, $t \in K$, for all $K \subseteq \mathbb{R}$ compact.

Theorem 20. [ADSZ, Theorem 9(3)](*LLT- Continuous version*) Assume that $\varphi : X \rightarrow \mathbb{R}$ is aperiodic and I is a bounded interval. Then

$$\sigma\sqrt{n}\hat{T}_{\beta}^n(\mathbb{1}_{\{\varphi_n \in k_n + I\}})(x) \xrightarrow{n \rightarrow \infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}(x), \quad k_n \in \mathbb{R}, \quad \frac{k_n - nE_m(\varphi)}{\sigma\sqrt{n}} \rightarrow t$$

uniformly in $x \in X$, $t \in K$, for all $K \subseteq \mathbb{R}$ compact.

Remark 21. Uniformity in t in the above theorems should be interpreted as follows: Let $K \subseteq \mathbb{R}$ be compact and assume that for every $t \in K$ we have a sequence $k_n(t)$ such that $\frac{k_n(t) - nE_m(\varphi)}{\sigma\sqrt{n}}$ converges to t uniformly as $n \rightarrow \infty$, then

$$\sigma\sqrt{n}\hat{T}_{\beta}^n(\mathbb{1}_{\{\varphi_n \in k_n(t) + I\}})(x) \xrightarrow{n \rightarrow \infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}$$

uniformly in $x \in X$, $t \in K$.

The following theorems are a version of the two previous ones that instead of giving actual limits provide an upper bound for $\hat{T}_{\beta}^n(\mathbb{1}_{\{\varphi_n = k\}})$ and $\hat{T}_{\beta}^n(\mathbb{1}_{\{\varphi_n \in I + y\}})$ for all $k \in \mathbb{Z}$, $y \in \mathbb{R}$. The proof is essentially the same as the proofs of the LLT theorem.

Theorem 22. (*Discrete version*) Assume that $\varphi : X \rightarrow \mathbb{Z}$ is aperiodic. Then there exists a constant C such that

$$\hat{T}_{\beta}^n(\mathbb{1}_{\{\varphi_n = k\}})(x) \leq \frac{C}{\sqrt{n}}$$

for all $k \in \mathbb{Z}$, $x \in X$.

Theorem 23. (*Continuous version*) Assume that $\varphi : X \rightarrow \mathbb{R}$ is aperiodic and $I \subseteq \mathbb{R}$ is a bounded interval. Then there exists a constant C such that

$$\hat{T}_{\beta}^n(\mathbb{1}_{\{\varphi_n \in I + y\}})(x) \leq \frac{C}{\sqrt{n}}$$

for all $y \in \mathbb{R}$, $x \in X$.

5. ASYMPTOTIC DISTRIBUTIONAL STABILITY

As explained in the introduction our objective is to prove asymptotic distributional stability for the random walk adic transformation. This is the goal of the present section.

Let $G = \mathbb{Z}$ or $G = \mathbb{R}$ and let $\varphi : X \rightarrow \mathbb{R}$ be aperiodic with $C_{\varphi, \alpha} < \infty$. As explained in the previous section, in this case, the random walk adic transformation

$$\left(\hat{X} \times G, \left(\mathcal{B} \cap \hat{X} \right) \times \mathcal{B}(G), \mu, \tau_{\varphi} \right)$$

is conservative and ergodic. Let χ be a standard Gaussian random variable defined on some probability space. For two random variable Y and Z we write $Y \stackrel{d}{=} Z$ if Y has the same distribution as Z . We prove

Theorem 24. *The random walk adic transformation is distributionally stable with return sequence $a_n \propto \frac{n}{\sqrt{\log n}}$ and a random variable $Y \stackrel{d}{=} e^{-\chi^2}$ in the limit, i.e*

$$(5.1) \quad \frac{1}{a_n} S_n(f) \xrightarrow{\mathcal{L}^{(m)}} e^{-\chi^2} m(f)$$

for all $f \in L^1(m)$, $f \geq 0$, where $S_n(f) := \sum_{k=0}^{n-1} f \circ \tau_{\varphi}^k$.

Remark 25. The theorem is valid for an aperiodic random walk on $G = \mathbb{Z}^k \times \mathbb{R}^{D-k}$ with return sequence $a_n \propto \frac{n}{(\log n)^{\frac{D}{2}}}$. The changes needed for the proof in this setting are statements of theorems in section (4) for $G = \mathbb{Z}^k \times \mathbb{R}^{D-k}$ as in [AS]. In this case the random variable $e^{-\chi_D^2}$ appears in the limit, where $\chi_D^2 = \|\xi^2\|_2^2$ for ξ a standard Gaussian random vector in \mathbb{R}^D .

5.1. Overview of the proof. Similarly to the methods of [AS] we split the τ orbit up to time n of a point $x \in \hat{X}$ into smaller blocks, where each block is of the form $\{T^{-l_n} \tau^i T^{l_n} x\}$ with $l_n \propto \log n$. Since the topological entropy of T_{β} is $\log \beta$, each block is roughly of size β^{l_n} (lemma 26). Over these blocks we are able to estimate the sums $S_n(f)(x, y)$ for $f = \mathbb{1}_{\hat{X} \times I}$ where I is a bounded interval using the LLT (lemma 29). This will allow us to prove that

(5.1) holds for $f = \mathbb{1}_{\hat{X} \times I}$, which by Hopf's ergodic theorem is sufficient to obtain theorem 24 (section 5.4).

5.2. Conventions and notations. Throughout the remaining part of this paper we use the following conventions:

- (1) For $a, b \in \mathbb{R}$, $c > 0$ we write $a = b \pm c$ if $a \leq b + c$ and $a \geq b - c$.
- (2) For $I \subseteq G$, $|I|$ denotes the Haar measure of I (we use this in order to distinguish between the Lebesgue measure on \hat{X} which we denote by λ and the Haar measure on $G = \mathbb{R}$ or $G = \mathbb{Z}$).
- (3) $S_n(f)(x, y) := \sum_{k=0}^{n-1} f(\tau_\varphi^k(x, y))$
- (4) $\phi_n := \sum_{k=0}^{n-1} \phi(\tau^k x)$
- (5) $E_m(f) := \int_X f(x) dm(x)$; $Var_m(f) = E_m(f^2) - E_m^2(f)$
- (6) For $x \sim_{\mathcal{T}(T_\beta)} x'$, set $N(x, x') := \min \left\{ n \in \mathbb{N} : x_j = x'_j \forall j \geq n \right\}$

5.3. Estimates. Set $J_n(x) = \# \{y : y \in T^{-n}(x)\}$.

For n fixed, we call a point $x \in \hat{X}$

- n -minimal if $x = \min \left\{ T_\beta^{-n} \left(T_\beta^n x \right) \right\}$ and
- n -maximal if $x = \max \left\{ T_\beta^{-n} \left(T_\beta^n x \right) \right\}$;

where the minimum and the maximum are with respect to the reverse lexicographic order.

Define $K_n : \hat{X} \rightarrow \mathbb{N}$ and $\tau_n : \hat{X} \rightarrow \hat{X}$ by

$$K_n(x) = \min \left\{ k : \tau^k(x) \text{ is } n \text{ maximal} \right\}$$

and $\tau_n(x) : \hat{X} \rightarrow \hat{X}$ by

$$\tau_n(x) = \tau^{K_n(x)+1}.$$

Then

- $\tau_n(x)$ is n -minimal as it must have zeroes in the first n coordinates of its β -expansion (see section 2.2 for details on the structure of τ).

- $T^n(\tau_n x) = \tau(T^n x)$.
- $K_n(x) \leq J_n(x) := \#T^{-n}(T^n(x))$ with equality if x is n -minimal.

Set $K_n^r(x) := K_n(x) + \sum_{j=1}^{r-1} K_n(\tau_n^j(x))$ and $K_n^0 = 0$.

Eventually, as explained in 5.1, for every $x \in \hat{X}$ we approximate n by $K_{l_n}^r(x)$, where $l_n \sim \log_\beta n$ and $r = r_n(x)$ is large. This allows us to split the orbit of x under τ up to time n , into blocks of the form $T_\beta^{-l_n} \left(T_\beta^{l_n} \left(\tau_n^j(x) \right) \right)$ with cardinality of each block equal to $K_{l_n} \left(\tau_{l_n}^j(x) \right)$, $j = 1, \dots, r-1$. On each of these blocks, we are able to use the local limit theorem, in order to obtain a total estimate for $\mathbb{1}_{\hat{X} \times I} \left(S_{K_{l_n}^r}(x, 0) \right)$ where $I = \{0\}$ if $G = \mathbb{Z}$ and I is Riemann integrable with $|I| < \infty$ if $G = \mathbb{R}$.

We start with a lemma that provides an estimate of $K_n^r(x)$ on a large set of $x \in \hat{X}$. Note that the set depends on n , but not on r , if r is large enough.

Lemma 26. *For every $\epsilon > 0$ there exist $R, N \in \mathbb{N}$ such that for every $n > N$, there exists a set A_n^R with $\lambda(A_n^R) \geq 1 - \epsilon$, such that for every $r > R$ and $x \in A_n^R$, $K_n^r(x) = \beta^n r (1 \pm \epsilon)$.*

Proof. Fix $\epsilon > 0$. We have

$$\begin{aligned}
K_n^r(x) &= K_n(x) + \sum_{j=1}^{r-1} K_n(\tau_n^j x) \\
&= K_n(x) + \sum_{j=1}^{r-1} J_n(\tau_n^j x) \\
&= K_n(x) + \sum_{j=1}^{r-1} \# \left\{ T_\beta^{-n} (T_\beta^n \tau_n^j x) \right\} \\
&= K_n(x) + \sum_{j=1}^{r-1} \# \left\{ T_\beta^{-n} \tau^j T_\beta^n x \right\}
\end{aligned}$$

where the last equality follows from $T^n(\tau_n x) = \tau(T^n x)$.

By definition of the transfer operator \mathcal{L} (see section 4),

$$\begin{aligned}
\# \{T^{-n} \tau^j T^n x\} &= (\mathcal{L}_n \mathbb{1}) (\tau^j T_\beta^n x) \\
&= \beta^n h(\tau^j T_\beta^n x) \hat{T}_\beta^n \left(\frac{1}{h} \right) (\tau^j T_\beta^n x) \\
&= \beta^n h(\tau^j T_\beta^n x) \left(E_m \left(\frac{1}{h} \right) \pm \eta^n \right) \\
&= \beta^n (h(\tau^j T_\beta^n x) \pm \eta^n)
\end{aligned}$$

where $0 < \eta < 1$. It follows that

$$K_n^r(x) = K_n(x) + \beta^n \left(\sum_{j=1}^{r-1} h(\tau^j T_\beta^n x) \pm r\eta^n \right).$$

A similar computation gives $K_n(x) \leq \#T^{-n}(T^n x) \leq C\beta^n$ where C is some constant.

Set $A_n^R(\epsilon) := \left\{ x : \sum_{j=1}^{r-1} h(\tau^j T^n x) = r \pm \frac{\epsilon}{2} \forall r > R \right\}$ and consider the set $A_1^R(\epsilon)$. Since $E_\lambda(h) = 1$, by the ergodic theorem we have that $\lambda(A_1^R(\epsilon)) \geq 1 - \epsilon$ if R is large enough. Since every measurable $A \subseteq X$ satisfies $\left(1 - \frac{1}{\beta}\right) \lambda(A) \leq m(A) \leq \frac{1}{1-\beta} \lambda(A)$, it follows that there exists $R > 1$ such that for every $n \in \mathbb{N}$, $m(A_n^R(\epsilon)) = m(T^{-n} A_1^R(\epsilon)) = m(A_1^R(\epsilon)) \geq 1 - \epsilon$. Therefore, there exists R' such that for every $n \in \mathbb{N}$, $r > R'$, $\lambda(A_n^r(\epsilon)) \geq 1 - \epsilon$. Let $R'', N \in \mathbb{N}$ be such that $\forall n > N$, $r > R''$, we have $\eta^n < \frac{\epsilon}{4}$ and $\frac{C}{r} < \frac{\epsilon}{4}$. Then for $n > N$, $r > \max(R', R'')$, $x \in A_n^R := A_n^{\max(R', R'')}(\epsilon)$ we have

$$\begin{aligned}
K_n^r(x) &\leq K_n(x) + \beta^n \left(\sum_{j=1}^{r-1} h(\tau^j T_\beta^n x) + r\eta^n \right) \\
&\leq \beta^n r \left(1 + \frac{C}{r} + \eta^n + \frac{\epsilon}{2} \right) \\
&< \beta^n r (1 + \epsilon)
\end{aligned}$$

and similarly

$$\begin{aligned} K_n^r(x) &\geq \beta^n \left(\sum_{j=1}^{r-1} h(\tau^j T_\beta^n x) - r\eta^n \right) \\ &> \beta^n r (1 - \epsilon). \end{aligned}$$

The result follows from this. \square

Lemma 27. *For $I \subseteq G$ measurable,*

$$\begin{aligned} &S_{K_n^r(x)} \left(\mathbb{1}_{\hat{X} \times I} \right) (x, y) \\ &\leq \sum_{j=0}^{r-1} \# \left\{ z \in T_\beta^{-n} (\tau^j T_\beta^n (x)) : \sum_{k=0}^{n+N(T^n x, \tau^j T^n x)} \varphi(T_\beta^k x) - \varphi(T_\beta^k(z)) \in I - y \right\} \end{aligned}$$

and

$$\begin{aligned} &S_{K_n^r(x)} \left(\mathbb{1}_{\hat{X} \times I} \right) (x, y) \\ &\geq \sum_{j=1}^{r-1} \# \left\{ z \in T_\beta^{-n} (\tau^j T_\beta^n (x)) : \sum_{k=0}^{n+N(T^n x, \tau^j T^n x)} \varphi(T_\beta^k x) - \varphi(T_\beta^k(z)) \in I - y \right\}. \end{aligned}$$

Proof. By definition

$$\begin{aligned} &S_{K_n^r(x)} \left(\mathbb{1}_{\hat{X} \times I} \right) (x, 0) \\ (5.2) \quad &= S_{K_n(x)} \left(\mathbb{1}_{\hat{X} \times I} \right) (x, y) + \sum_{j=1}^{r-1} S_{K_n^{j+1}(x)} \left(\mathbb{1}_{\hat{X} \times I} \right) (x, y) - S_{K_n^j(x)} \left(\mathbb{1}_{\hat{X} \times I} \right) (x, y). \end{aligned}$$

For fixed $j \geq 1$,

$$\begin{aligned}
S_{K_n^{j+1}(x)} \left(\mathbb{1}_{\hat{X} \times I} \right) (x, y) &= S_{K_n^j(x)} \left(\mathbb{1}_{\hat{X} \times I} \right) (x, y) \\
&= \sum_{l=K_n^j(x)}^{K_n^{j+1}(x)-1} \mathbb{1}_{\hat{X} \times I} \left(\tau^l x, y + \phi_l(x) \right) \\
&= \sum_{l=0}^{K_n(\tau_n^j(x))-1} \mathbb{1}_{\hat{X} \times I} \left(\tau^l \left(\tau^{K_n^j(x)} x \right), y + \phi_{K_n^{(j)}(x)+l}(x) \right) \\
&= \sum_{l=0}^{J_n(\tau_n^j(x))-1} \mathbb{1}_{\hat{X} \times I} \left(\tau^l \left(\tau^{K_n^j(x)} x \right), y + \phi_{K_n^{(j)}(x)+l}(x) \right).
\end{aligned}$$

Now by the properties listed in the beginning of this section,

$$\left\{ \tau^l \left(\tau^{K_n^j(x)}(x) \right) : l = 0, \dots, J_n(\tau_n^j(x)) - 1 \right\} = T_\beta^{-n} \left(T_\beta^n \left(\tau^{K_n^j(x)} x \right) \right) = T_\beta^{-n} \left(\tau^j \left(T_\beta^n x \right) \right).$$

Moreover, since for $x \sim_{\mathcal{T}(T_\beta)} x'$, $N(x, x') = \min \left\{ n \in \mathbb{N} : x_j = x'_j \forall j \geq n \right\}$, for $M > 0$, we have

$$\begin{aligned}
\phi_M(x) &= \sum_{i=0}^{M-1} \phi(\tau^i x) \\
&= \sum_{i=0}^{M-1} \psi(\tau^i x, \tau^{i+1} x) \\
&= \sum_{i=0}^{M-1} \sum_{k=0}^{\infty} \varphi \left(T_\beta^k(\tau^i x) \right) - \varphi \left(T_\beta^k(\tau^{i+1} x) \right) \\
&= \sum_{i=0}^{M-1} \sum_{k=0}^{N(\tau^i x, \tau^{i+1} x)} \varphi \left(T_\beta^k(\tau^i x) \right) - \varphi \left(T_\beta^k(\tau^{i+1} x) \right) \\
&= \sum_{i=0}^{M-1} \sum_{k=0}^{N(x, \tau^M x)} \varphi \left(T_\beta^k(\tau^i x) \right) - \varphi \left(T_\beta^k(\tau^{i+1} x) \right)
\end{aligned}$$

$$= \sum_{k=0}^{N(x, \tau^M x)} \varphi \left(T_\beta^k x \right) - \varphi \left(T_\beta^k (\tau^M x) \right)$$

where the one prior to the last equality follows because $N(\tau^i x, \tau^{i+1} x) \leq N(x, \tau^M x)$ for $i \leq M-1$ and the extra terms in the sum vanish. It follows that

$$\begin{aligned} & J_n \left(\tau_n^{(j)}(x) \right) - 1 \\ & \sum_{l=0}^{J_n \left(\tau_n^{(j)}(x) \right) - 1} \mathbb{1}_{\{\hat{X} \times I\}} \left(\tau^l \left(\tau^{K_n^{(j)}(x)} x \right), y + \phi_{K_n^{(j)}(x)+l}(x) \right) \\ & = \# \left\{ z \in T_\beta^{-n} \left(\tau^j T_\beta^n(x) \right) : \sum_{k=0}^{N(x, z)} \varphi \left(T_\beta^k x \right) - \varphi \left(T_\beta^k(z) \right) \in I - y \right\}. \end{aligned}$$

Similarly, since $K_n(x) \leq J_n(x)$,

$$\begin{aligned} S_{K_n(x)} \left(\mathbb{1}_{\hat{X} \times I} \right) (x, y) &= \sum_{l=0}^{K_n(x)-1} \mathbb{1}_{\hat{X} \times I} \left(\tau^l(x), y + \phi_l(x) \right) \\ &\leq \sum_{l=0}^{J_n(x)} \mathbb{1}_{\hat{X} \times I} \left(\tau^l(x), y + \phi_l(x) \right) \\ &= \# \left\{ z \in T_\beta^{-n} (T_\beta^n x) : \sum_{k=0}^{N(x, z)} \varphi \left(T_\beta^k x \right) - \varphi \left(T_\beta^k(z) \right) \in I - y \right\} \\ &= \# \left\{ z \in T_\beta^{-n} (T_\beta^n x) : \sum_{k=0}^n \varphi \left(T_\beta^k x \right) - \varphi \left(T_\beta^k(x) \right) \in I - y \right\}, \end{aligned}$$

where the last equality follows since for $z \in T_\beta^{-n} (T_\beta^n x)$, $N(x, z) \leq n$ and the extra terms in the sum vanish.

Note that for $j \geq 1$, $z \in T_\beta^{-n} (\tau^j T_\beta^n(x))$,

$$N(x, z) = n + N(T_\beta^n x, T_\beta^n z) = n + N(T_\beta^n x, \tau^j T_\beta^n x).$$

Thus

$$\begin{aligned} & \# \left\{ z \in T_\beta^{-n} (\tau^j T_\beta^n (x)) : \sum_{k=0}^{N(x,z)} \varphi(T_\beta^k x) - \varphi(T_\beta^k (z)) \in I - y \right\} \\ &= \# \left\{ z \in T_\beta^{-n} (\tau^j T_\beta^n (x)) : \sum_{k=0}^{n+N(T_\beta^n x, \tau^j T_\beta^n x)} \varphi(T_\beta^k x) - \varphi(T_\beta^k (z)) \in I - y \right\} \end{aligned}$$

whence the lemma is proved by summing over j and dropping the term $S_{K_n(x)}$ from the sum in (5.2) for the lower bound. \square

The next lemma shows that $\max_{0 \leq j \leq r} N(T_\beta^n x, \tau^j T_\beta^n x)$ is negligible compared to n , for all r bounded by some constant. This is used in lemma 29 for estimating sums of the type $\sum_{k=0}^{n+N(T_\beta^n x, \tau^j T_\beta^n x)} \varphi(T_\beta^k x) - \varphi(T_\beta^k (z))$ using the LLT.

Lemma 28. *Let $C > 0$. Then for all $r < C$, and $M > C \log \beta$ the set $D_n^r(M) := \{x \in \hat{X} : \max_{1 \leq j \leq r} N(T_\beta^n x, \tau^j T_\beta^n x) \geq M \log n\}$ has Lebesgue measure 0 if n is large enough.*

Proof. Since the quantity $N(x, \tau^j x)$ increases as j increases, we have

$$\begin{aligned} C_n(M) &\subseteq \{x : N(T_\beta^n x, \tau^r (T_\beta^n x)) \geq M \log n\} \\ &\subseteq \left\{ x : \max_{0 \leq i \leq r-1} N(\tau^i T_\beta^n x, \tau^{i+1} T_\beta^n x) \geq \frac{M \log n}{r} \right\}. \end{aligned}$$

Since τ preserves the Lebesgue measure,

$$\begin{aligned} \lambda \left\{ x : \max_{0 \leq i \leq r} N(\tau^i x, \tau^{i+1} x) \geq \frac{M \log n}{r} \right\} &= \lambda \left\{ \bigcup_{i=0}^{r-1} \left\{ x : N(\tau^i x, \tau^{i+1} x) \geq \frac{M \log n}{r} \right\} \right\} \\ &\leq r \lambda \left\{ x : N(x, \tau x) \geq \frac{M \log n}{r} \right\}. \end{aligned}$$

Since $N(x, \tau x) \geq \frac{M \log n}{r}$ implies that $T^i x \geq 1 - \frac{1}{\beta}$ for all $i \leq \frac{M \log n}{r}$ (see proposition 8 and the proof therein), we have $\lambda \left\{ x : N(x, \tau x) \geq \frac{M \log n}{r} \right\} \leq C' \frac{1}{\beta \frac{M \log n}{r}}$ where C' is some constant.

Since the sum $\sum_{n=1}^{\infty} \beta^{-\frac{M \log n}{r}}$ converges if $\frac{M}{r} > \log \beta$, it follows by the Borel-Cantelli lemma that $\lambda(D_n^r(M)) = 0$ if n is large enough. \square

Lemma 29. *Let $I = \{0\}$ if $G = \mathbb{Z}$ and $I \subseteq \mathbb{R}$ a bounded interval if $G = \mathbb{R}$ and let C, δ be some positive constant,. Then for all $\epsilon > 0, r < C$ there exists N such that for all $n > N$, $x \in \hat{X}, y \in I$,*

$$\begin{aligned} & \mathbb{1}_{B(0,\delta)} \left(\frac{\bar{\varphi}_n(x)}{\sqrt{n}} \right) S_{K_n^r(x)} \left(\mathbb{1}_{\hat{X} \times I} \right) (x, y) \\ & \leq \mathbb{1}_{B(0,\delta)} \left(\frac{\bar{\varphi}_n(x)}{\sqrt{n}} \right) \left(\frac{|I| \beta^n \sum_{j=0}^{r-1} h(\tau^j T_\beta^n x)}{\sigma \sqrt{2\pi n}} \left(e^{-\frac{\bar{\varphi}_n^2(x)}{2\sigma^2 n}} + \epsilon \right) \right) \end{aligned}$$

and

$$\begin{aligned} & \mathbb{1}_{B(0,\delta)} \left(\frac{\bar{\varphi}_n(x)}{\sqrt{n}} \right) S_{K_n^r(x)} \left(\mathbb{1}_{\hat{X} \times I} \right) (x, y) \\ & \geq \mathbb{1}_{B(0,\delta)} \left(\frac{\bar{\varphi}_n(x)}{\sqrt{n}} \right) \left(\frac{|I| \beta^n \sum_{j=1}^{r-1} h(\tau^j T_\beta^n x)}{\sigma \sqrt{2\pi n}} \left(e^{-\frac{\bar{\varphi}_n^2(x)}{2\sigma^2 n}} - \epsilon \right) \right) \end{aligned}$$

where $\sigma^2 = \frac{1}{n} \text{Var}_m \left(\sum_{k=0}^n \varphi \circ T_\beta^k \right)$.

Proof. By lemma 26

$$\begin{aligned} & S_{K_n^r(x)} \left(\mathbb{1}_{\hat{X} \times I} \right) (x, y) \\ & \leq \sum_{j=0}^{r-1} \# \left\{ z \in T_\beta^{-n} (\tau^j T_\beta^n(x)) : \sum_{k=0}^{n+N(T^n x, \tau^j T^n x)} \varphi(T_\beta^k x) - \varphi(T_\beta^k(z)) \in I - y \right\} \end{aligned}$$

and

$$S_{K_n^r(x)} \left(\mathbb{1}_{\hat{X} \times I} \right) (x, y)$$

$$\geq \sum_{j=1}^r \# \left\{ z \in T_{\beta}^{-n} (\tau^j T_{\beta}^n(x)) : \sum_{k=0}^{n+N(T^n x, \tau^j T^n x)} \varphi(T_{\beta}^k x) - \varphi(T_{\beta}^k(z)) \in I - y \right\}.$$

Now for fixed j , setting $k_n(x, z) = \sum_{k=n+1}^{N(T^n x, \tau^j T^n x)} \varphi(T_{\beta}^k x) - \varphi(T_{\beta}^k z)$ we have,

$$\begin{aligned} & \# \left\{ z \in T_{\beta}^{-n} (\tau^j T_{\beta}^n(x)) : \sum_{k=0}^{n+N(T^n x, \tau^j T^n x)} \varphi(T_{\beta}^k x) - \varphi(T_{\beta}^k(z)) \in I \right\} \\ &= \sum_{z \in T^{-n} \tau^j T^n x} \mathbb{1}_{\{\varphi_n(z) \in \varphi_n(x) + k_n(x, z) + I - y\}} \\ &= \mathcal{L}^n \left(\mathbb{1}_{\{\varphi_n(\cdot) \in \varphi_n(x) + k_n(x, \cdot) + I - y\}} \right) (\tau^j T^n x) \\ &= \beta^n h(\tau^j T^n x) \hat{T}^n \left(\frac{\mathbb{1}_{\{\varphi_n(\cdot) \in \varphi_n(x) - k_n(x, \cdot) + I - y\}}}{h(\cdot)} \right) (\tau^j T^n x). \end{aligned}$$

Since r is bounded, by lemma 28 if n is large enough $k_n(x, z) \leq M \sup |\varphi| \log n$, where M is constant, and therefore by LLT, there exists N , such that for all $n > N$, $x \in \hat{X}$, $y \in I$ we have

$$\begin{aligned} & \mathbb{1}_{B(0, \delta)} \left(\frac{\bar{\varphi}_n}{\sqrt{n}} \right) \beta^n h(\tau^j T^n x) \hat{T}^n \left(\frac{\mathbb{1}_{\{\varphi_n(\cdot) \in \varphi_n(x) - k_n(x, \cdot) + I - y\}}}{h(\cdot)} \right) (\tau^j T^n x) \\ &= \mathbb{1}_{B(0, \delta)} \left(\frac{\bar{\varphi}_n}{\sqrt{n}} \right) \beta^n h(\tau^j T^n x) \frac{|I|}{\sigma \sqrt{2\pi n}} \left(e^{-\frac{\bar{\varphi}_n^2(x)}{2\sigma^2 l_n}} \pm \epsilon \right). \end{aligned}$$

whence the lemma is proved by summing over j . \square

Lemma 30. *Let I be as in lemma 29. For every $\epsilon > 0$, there exists $K \in \mathbb{N}$, such that for every $n > K$ there exists a set A_n with $\lambda(A_n) \geq 1 - \epsilon$, so that for every $x \in A_n$, $y \in I$*

$$\mathbb{1}_{B(0, \delta)} \left(\frac{\bar{\varphi}_n}{\sqrt{n}} \right) \frac{\sqrt{l_n}}{n} S_n \left(\mathbb{1}_{\hat{X} \times I} \right) (x, y) = |I| \frac{1}{\sigma \sqrt{2\pi}} \left(e^{-\frac{\bar{\varphi}_{l_n}^2(x)}{2\sigma^2 l_n}} \pm \epsilon \right)$$

where $l_n \sim \log_{\beta} n$, $\delta > 0$.

Proof. Fix $\epsilon > 0$. Let N, R be as in lemma 26. Set $l_n = \left\lceil \log_{\beta} \left(\frac{n}{(R+2)(1+\epsilon)} \right) \right\rceil$. Let n be large enough so that $l_n > N$. Then by lemma 26 there exists a set A_n with $\lambda(A_n) > 1 - \epsilon$, such

that for all $r > R$, and all $x \in A_n$,

$$K_{l_n}^r(x) = \beta^{l_n} r (1 \pm \epsilon).$$

For $x \in A_n$, let $r_n(x)$ be such that $K_{l_n}^{r_n(x)}(x) \leq n < K_{l_n}^{r_n(x)+1}$. Since for $r \leq R$,

$$K_{l_n}^{r+1}(x) \leq K_{l_n}^{R+1}(x) = \beta^{l_n} (R+1) (1 \pm \epsilon) \leq \frac{n}{(R+2)} (R+1) < n$$

it follows that $r_n(x) > R$. Moreover, since

$$n \geq K_{l_n}^{r_n(x)} \geq \beta^{l_n} r_n(x) (1 - \epsilon) \geq \left(\frac{n}{(R+2)(1+\epsilon)} - \beta \right) (r_n(x)) (1 - \epsilon)$$

we have $r_n(x) \leq C$ where C depends only on R, ϵ, β . By lemma 29 there exists N' such that

for all $n > N'$, $r \leq C$, $x \in \hat{X}$, $y \in I$

$$\mathbb{1}_{B(0,\delta)} \left(\frac{\bar{\varphi}_n(x)}{\sqrt{n}} \right) S_{K_n^r(x)} \left(\mathbb{1}_{\hat{X} \times I} \right) (x, y) \leq \mathbb{1}_{B(0,\delta)} \left(\frac{\bar{\varphi}_n(x)}{\sqrt{n}} \right) \left(|I| \frac{\beta^n \sum_{j=0}^{r-1} h(\tau^j T_\beta^n x)}{\sqrt{n}} \left(e^{-\frac{\bar{\varphi}_n(x)}{2n}} + \epsilon \right) \right)$$

and

$$\mathbb{1}_{B(0,\delta)} \left(\frac{\bar{\varphi}_n(x)}{\sqrt{n}} \right) S_{K_n^r(x)} \geq \mathbb{1}_{B(0,\delta)} \left(\frac{\bar{\varphi}_n(x)}{\sqrt{n}} \right) \left(|I| \frac{\beta^n \sum_{j=1}^{r-1} h(\tau^j T_\beta^n x)}{\sqrt{n}} \left(e^{-\frac{\bar{\varphi}_n(x)}{2n}} - \epsilon \right) \right)$$

It follows that for n such that $l_n > \max(N, N')$, and for all $x \in A_n$,

$$\begin{aligned} S_n \left(\mathbb{1}_{\hat{X} \times I} \right) (x, y) &\geq S_{K_{l_n}^{r_n(x)}(x)} \left(\mathbb{1}_{\hat{X} \times I} \right) (x, y) \\ &\geq \frac{\beta^{l_n} \sum_{j=1}^{r_n(x)-1} h(\tau^j T^n x)}{\sigma \sqrt{2\pi l_n}} \left(e^{-\frac{\bar{\varphi}_{l_n}(x)}{2l_n}} - \epsilon \right) \\ &\geq \frac{\beta^{l_n} r_n(x) (1 - \epsilon)}{\sigma \sqrt{2\pi l_n}} \left(e^{-\frac{\bar{\varphi}_{l_n}(x)}{2l_n}} - \epsilon \right) \end{aligned}$$

where in the last inequality we use $\sum_{j=1}^{r_n(x)-1} h(\tau^j T^n x) \geq r_n(x)(1-\epsilon)$ for $r > R$, which we may assume to be true by the proof of lemma 26. Similarly

$$\begin{aligned} S_n \left(\mathbb{1}_{\hat{X} \times I} \right) (x, 0) &\leq S_{K_{l_n}^{r_n(x)+1}(x)} \left(\mathbb{1}_{\hat{X} \times I} \right) (x, 0) \\ &\leq \frac{\beta^{l_n} \sum_{j=0}^{r_n(x)} h(\tau^j T^n x)}{\sigma \sqrt{2\pi l_n}} \left(e^{-\frac{\bar{\varphi}_{l_n}(x)}{2l_n}} + \epsilon \right) \\ &\leq \frac{\beta^{l_n} r_n(x)(1+\epsilon)}{\sigma \sqrt{2\pi l_n}} \left(e^{-\frac{\bar{\varphi}_{l_n}(x)}{2l_n}} + \epsilon \right). \end{aligned}$$

Since $K_{l_n}^{r_n(x)} \leq n \leq K_{l_n}^{r_n(x)+1}$ we have

$$n \geq K_{l_n}^{r_n(x)} \geq \beta^{l_n} (r_n(x) - \epsilon)$$

and

$$n \leq K_{l_n}^{r_n(x)+1} \leq \beta^{l_n} (r_n(x) + 1 + \epsilon).$$

It follows that

$$\frac{\beta^{l_n} r_n(x)}{n} \leq 1 + \frac{\beta^{l_n} \epsilon}{n} \leq 1 + \epsilon$$

and

$$\frac{\beta^{l_n} r_n(x)}{n} \geq 1 - \frac{1}{R+2} - \frac{1}{n}$$

and we may assume that $\frac{1}{R+2} < \epsilon$ by enlarging R if necessary. Thus,

$$\frac{\sqrt{l_n}}{n} S_n \left(\mathbb{1}_{\hat{X} \times I} \right) (x, 0) \leq \frac{(1+\epsilon)^2}{\sigma \sqrt{2\pi}} \left(e^{-\frac{\bar{\varphi}_{l_n}(x)}{2l_n}} + \epsilon \right)$$

and

$$\frac{\sqrt{l_n}}{n} S_n \left(\mathbb{1}_{\hat{X} \times I} \right) (x, 0) \geq \frac{(1-\epsilon)^2}{\sigma \sqrt{2\pi}} \left(e^{-\frac{\bar{\varphi}_{l_n}(x)}{2l_n}} - \epsilon \right)$$

and the lemma follows from this. \square

The following lemma will only be used in the proof of bounded rational ergodicity of the random walk adic transformation (see section 6).

Lemma 31. *Let I be as in lemma 29. There exists $C > 0$, such that for all $n, r \in \mathbb{N}$, $(x, y) \in \hat{X} \times G$,*

$$S_{K_n^r(x)} \left(\mathbb{1}_{\hat{X} \times I} \right) (x, y) \leq Cr \frac{\beta^n}{\sqrt{n}}.$$

Proof. By lemma 27

$$\begin{aligned} S_{K_n^r(x)} \left(\mathbb{1}_{\hat{X} \times I} \right) (x, y) \\ \leq \sum_{j=0}^{r-1} \# \left\{ z \in T_\beta^{-n} \left(\tau^j T_\beta^n(x) \right) : \sum_{k=0}^{n+N(T^n x, \tau^j T^n x)} \varphi \left(T_\beta^k x \right) - \varphi \left(T_\beta^k(z) \right) \in I - y \right\} \end{aligned}$$

Similarly to the calculation in the proof of lemma 29, setting

$$k_n(x, z) = \sum_{k=n+1}^{N(T^n x, \tau^j T^n x)} \varphi \left(T_\beta^k x \right) - \varphi \left(T_\beta^k z \right)$$

we have

$$\begin{aligned} \sum_{j=0}^{r-1} \# \left\{ z \in T_\beta^{-n} \left(\tau^j T_\beta^n(x) \right) : \sum_{k=0}^{n+N(T^n x, \tau^j T^n x)} \varphi \left(T_\beta^k x \right) - \varphi \left(T_\beta^k(z) \right) \in I - y \right\} \\ = \sum_{j=0}^{r-1} \mathcal{L}^n \left(\mathbb{1}_{\{\varphi_n(\cdot) \in \varphi_n(x) + k_n(x, \cdot) + I - y\}} \right) (\tau^j T^n x) \\ = \sum_{j=0}^{r-1} \beta^n h \left(\tau^j T^n x \right) \hat{T}^n \left(\frac{\mathbb{1}_{\{\varphi_n(\cdot) \in \varphi_n(x) - k_n(x, \cdot) + I - y\}}}{h(\cdot)} \right) (\tau^j T^n x) \\ \leq C \cdot r \frac{\beta^n}{\sqrt{n}}. \end{aligned}$$

where C is some constant. The last inequality follows from theorems 22, 23 using $k_n(x, z) \leq M \log n$ (see lemma 28). \square

5.4. Proof of theorem 24. Let g be bounded and continuous on $[0, \infty]$ and let $f \in L^1(\lambda \times dy)$, $f \geq 0$. Our objective is to prove that for $a_n \propto \frac{n}{\sqrt{\log n}}$,

$$g(S_n f) dm \longrightarrow Eg\left(e^{-\frac{1}{2}\chi^2}\right)$$

where χ is a standard Gaussian random variable.

Fix $\epsilon > 0$. Since we have $\frac{1}{n}Var_m\left(\sum \varphi \circ T_\beta^i\right) \longrightarrow \sigma^2 > 0$, it follows from Chebychev's inequality that if δ is large enough $m\left(\frac{\bar{\varphi}_n}{\sqrt{n}} \in B(0, \delta)\right) > 1 - \epsilon$ for all $n \in \mathbb{N}$. By the fact that the invariant density h is bounded from above and is bounded away from zero, the same is true with the Lebesgue measure λ instead of m . Thus, for every $n \in \mathbb{N}$, there exist a set B_n with $\lambda(B_n) > 1 - \epsilon$ such that $\frac{\bar{\varphi}_n(x)}{\sqrt{n}} \in B(0, \delta)$ for all $x \in B_n$. By lemma 30 there exists K , $l_n \sim \log n$, such that for all $n > K$, there exists a set A_n with $\lambda(A_n) > 1 - \epsilon$, so that

$$\mathbb{1}_{B(0, \delta)} \frac{\sqrt{l_n}}{n} S_n \left(\mathbb{1}_{\hat{X} \times I} \right) (x, y) = \mathbb{1}_{B(0, \delta)} \frac{|I|}{\sigma \sqrt{2\pi}} \left(e^{-\frac{\bar{\varphi}_{l_n}^2(s)}{2\sigma l_n}} \pm \epsilon \right)$$

for every $x \in A_n$, $y \in I$. By the uniform continuity of the function g this implies that on the set A_n ,

$$g \left(\mathbb{1}_{B(0, \delta)} \frac{\sqrt{l_n}}{n} S_n \left(\mathbb{1}_{\hat{X} \times I} \right) (x, y) \right) = g \left(\mathbb{1}_{B(0, \delta)} \frac{|I|}{\sigma \sqrt{2\pi}} e^{-\frac{\bar{\varphi}_{l_n}^2(s)}{2\sigma l_n}} \right) \pm \epsilon$$

Since $\lambda(A_n \cap B_n) > 1 - 2\epsilon$ it follows that

$$\left| \int_{\hat{X} \times I} g \left(\frac{\sqrt{l_n}}{n} S_n \left(\mathbb{1}_{\hat{X} \times I} \right) (x, y) \right) dm - \int_{\hat{X} \times I} g \left(\frac{|I|}{\sigma \sqrt{2\pi}} e^{-\frac{\bar{\varphi}_{l_n}^2(s)}{2\sigma l_n}} \right) dm \right| \leq \epsilon + 4\epsilon \sup |g|.$$

Now by the CLT,

$$\int_{\hat{X} \times I} g \left(\frac{|I|}{\sigma \sqrt{2\pi}} e^{-\frac{\bar{\varphi}_{l_n}^2(s)}{2\sigma l_n}} \right) dm \longrightarrow \int_I E \left(g \left(\frac{|I|}{\sigma \sqrt{2\pi}} e^{-\chi^2} \right) \right) dy = |I| E \left(g \left(\frac{|I|}{\sigma \sqrt{2\pi}} e^{-\chi^2} \right) \right).$$

It follows that there exists a sequence $a_n \propto \frac{n}{\sqrt{\log n}}$, such that for all bounded and continuous g on $[0, \infty]$, $p = \frac{\mathbb{1}_{\hat{X} \times I}}{m(\hat{X} \times I)} \in L^\infty(m)$, $h = \mathbb{1}_{\hat{X} \times G}$,

$$(5.3) \quad \int_{\hat{X} \times G} g \left(\frac{1}{a_n} S_n(h) \right) \cdot p dm \longrightarrow E \left(g \left(m(h) \cdot e^{-\chi^2} \right) \right).$$

We claim that this implies

$$(5.4) \quad \frac{1}{a_n} S_n(h) \xrightarrow{\mathcal{L}} e^{-\chi^2}.$$

To see this, assume by contradiction that this is not the case. Then by definition, there exists a probability measure $q \ll m$, a function $f \in C[0, \infty]$, $\epsilon > 0$ and a subsequence n_k such that

$$(5.5) \quad \left| \int_{\hat{X} \times G} f \left(\frac{1}{a_{n_k}} S_{n_k}(h) \right) \cdot q dm - E \left(f \left(m(h) \cdot e^{-\chi^2} \right) \right) \right| > \epsilon$$

for all $k \in \mathbb{N}$. By corollary 3.6.2 in [Aa1], there exists a further subsequence $m_l := n_{k_l}$ and a random variable Y on $[0, \infty]$, such that

$$\frac{1}{a_{m_l}} S_{m_l}(h) \xrightarrow{\mathcal{L}} Y.$$

It follows that for all $g \in C[0, \infty]$, and a probability measure $q \ll m$,

$$\int_{\hat{X} \times G} g \left(\frac{1}{a_{m_l}} S_{m_l}(h) \right) \cdot q dm \longrightarrow E \left(g \left(m(h) \cdot Y \right) \right).$$

But (5.3) implies that for $p = \frac{\mathbb{1}_{\hat{X} \times I}}{m(\hat{X} \times Y)}$, $g \in C[0, \infty]$,

$$\int_{\hat{X} \times G} g \left(\frac{1}{a_{m_l}} S_{m_l}(h) \right) \cdot p dm \longrightarrow E \left(g \left(m(h) e^{-\chi^2} \right) \right)$$

whence Y has the same distribution as $e^{-\chi^2}$. This contradicts (5.5) and therefore (5.4) holds.

As explained in the introduction, by Hopf's ergodic theorem ([Aa1, Corollary 3.6.2]), (5.4) implies $\frac{1}{a_n} S_n \xrightarrow{\nu} Y$, which proves the theorem.

6. BOUNDED RATIONAL ERGODICITY

In this section we prove:

Theorem 32. *The random walk adic transformation $(\hat{X} \times G, (\mathcal{B} \cap \hat{X}) \times \mathcal{B}(G), \mu, \tau_\varphi)$, with $\varphi : X \rightarrow G$ satisfying the assumptions of theorem 24 is bounded rationally ergodic with return sequence $a_n \propto \frac{\sqrt{\log n}}{n}$.*

Remark 33. As in the case of asymptotical distributional stability, the theorem is valid for an aperiodic random walk on $G = \mathbb{Z}^k \times \mathbb{R}^{D-k}$ with return sequence $a_n \propto \frac{n}{(\log n)^{\frac{D}{2}}}$. The changes needed for the proof in this setting are statements of theorems in section (4) for $G = \mathbb{Z}^k \times \mathbb{R}^{D-k}$ as in [AS].

Proof. Bounded rational ergodicity follows (see definition) if we prove that there exists a measurable $A \subseteq \hat{X} \times G$ and constants $C, c > 0$ such that

$$(6.1) \quad \int_A S_n(\mathbb{1}_A)(x, y) d\mu \geq \frac{cn}{\sqrt{\log n}}$$

and

$$(6.2) \quad \|S_n(\mathbb{1}_A)\|_\infty \leq \frac{Cn}{\sqrt{\log n}}.$$

Let $I = \{0\}$ in case $G = \mathbb{Z}$ and I a bounded interval in case $G = \mathbb{R}$. Fix $\epsilon > 0$. As in the proof of theorem 24 since $\frac{1}{n} \text{Var}_m \left(\sum \varphi \circ T_\beta^i \right) \rightarrow \sigma^2 > 0$, it follows from Chebychev's inequality that if δ is large enough then $m \left(\frac{\bar{\varphi}_n}{\sqrt{n}} \in B(0, \delta) \right) > 1 - \epsilon$ for all $n \in \mathbb{N}$. It follows that for every $n \in \mathbb{N}$, there exist a set B_n with $\lambda(B_n) > 1 - \epsilon$ such that $\frac{\bar{\varphi}_n(x)}{\sqrt{n}} \in B(0, \delta)$ for all $x \in B_n$. By lemma 30, there exists a set $A_n \subseteq \hat{X}$ with $\lambda(A_n) > 1 - \epsilon$, and a sequence $l_n \sim \log n$ such that

for every $x \in A_n$, $y \in I$,

$$\mathbb{1}_{B(0,\delta)} \left(\frac{\bar{\varphi}_n}{\sqrt{n}} \right) \frac{\sqrt{l_n}}{n} S_n \left(\mathbb{1}_{\hat{X} \times I} \right) (x, y) = |I| \frac{1}{\sigma \sqrt{2\pi}} \left(e^{-\frac{\bar{\varphi}_{l_n}^2(x)}{2\sigma^2 l_n}} \pm \epsilon \right).$$

It immediately follows from this that there exists $c > 0$, such that

$$\int_{\hat{X} \times I} S_n \left(\mathbb{1}_{\hat{X} \times I} \right) (x, y) d\mu \geq \int_{\hat{X} \times I} \mathbb{1}_{B_n \cap A_n} (x) S_n (\mathbb{1}) (x, y) d\mu \geq \frac{cn}{\sqrt{\log n}}$$

whence (6.1) is proved.

To prove (6.2) let $l_n := \lceil L \log_\beta n \rceil$ for some constant L to be chosen later and consider $K_{l_n}^2$. As in the proof of lemma 26 we have

$$K_{l_n}^2 (x) = K_{l_n} (x) + \beta^{l_n} h \left(\tau T_\beta^{l_n} x \right) \pm \eta^{l_n}$$

where $0 < \eta < 1$. Since $h \geq 1 - \frac{1}{\beta}$,

$$\beta^{l_n} h \left(\tau T_\beta^{l_n} x \right) \geq \frac{Ln}{\beta} \left(1 - \frac{1}{\beta} \right) \pm \eta^{l_n}.$$

It follows that there exists L such that $K_{l_n}^2 \geq n$ for all $n \in \mathbb{N}$. Thus, using lemma 31

$$\begin{aligned} S_n \left(\mathbb{1}_{\hat{X} \times I} \right) (x, y) &\leq S_{K_{l_n}^2(x)} \left(\mathbb{1}_{\hat{X} \times I} \right) \\ &\leq C \frac{\beta^{l_n}}{\sqrt{l_n}} \leq \tilde{C} \frac{n}{\sqrt{\log n}} \end{aligned}$$

whence 6.2 follows. □

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